Functional analysis
List 1

Ex 1. Check that spaces of *p-summable sequences*

$$l_p = \{x = (x(1), x(2), ...) : \|x\|_p = \left(\sum_{k=1}^{n} |x(k)|^p\right)^{\frac{1}{p}} < \infty\}, \qquad 1 \leq p < \infty,$$

are normed linear spaces (with $\|\cdot\|_p$ as a norm). Prove that the space of *bounded sequences*

$$l_\infty = \{x = (x(1), x(2), ...) : \|x\|_\infty = \sup_{k\in\mathbb{N}} |x(k)| < \infty\},$$

the space of *convergent sequences*

$$c = \{x = (x(1), x(2), ...) : \exists_g \lim_{k\to\infty} x(k) = g\},$$

and the space of *sequences convergent to zero*

$$c = \{x = (x(1), x(2), ...) : \lim_{k\to\infty} x(k) = 0\}$$

are normed linear spaces with $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x(k)|$ as a norm.

Ex 2. Prove that the spaces l_p, l_∞, c and c_0 are Banach spaces.

Ex 3. Show the inclusions

$$l_1 \subset ... \subset l_p \subset ... \subset l_q \subset ... \subset c_0 \subset c \subset l_\infty, \qquad 1 < p < q < \infty$$

and check that none of them may be replaced by equality.

Ex 4. Check, whether a sequence x_n of elements of X converges to an element a.

N	X	x_n	a
1.	l_1	$(\underbrace{\sin\frac{1}{2^n}, \sin\frac{1}{2^n}, ..., \sin\frac{1}{2^n}}_{n}, 0, ...)$	$(0, 0, ..., 0, ...)$
2.	l_3	$(\underbrace{\frac{n^2}{2^n}, \frac{n^2}{2^n}, ..., \frac{n^2}{2^n}}_{n}, 0, ...)$	$(1, 0, ..., 0, ...)$
3.	c	$(\underbrace{(\frac{4n+1}{4n+3})^n, (\frac{4n+1}{4n+3})^n, ..., (\frac{4n+1}{4n+3})^n}_{n}, 0, ...)$	$(e^{-\frac{1}{2}}, e^{-\frac{1}{2}}, ..., e^{-\frac{1}{2}}, ...)$
4.	$l_{\frac{8}{5}}$	$(\underbrace{\frac{\cos\frac{1}{n}}{n}, \frac{\cos\frac{1}{n}}{n}, ..., \frac{\cos\frac{1}{n}}{n}}_{n}, 0, ...)$	$(0, 0, ..., 0, ...)$
5.	l_∞	$(0, \frac{7}{8}, ..., \frac{n^3-1}{n^3}, 0, 0, ...)$	$(0, \frac{7}{8}, ..., \frac{k^3-1}{k^3}, \frac{(k+1)^3-1}{(k+1)^3}, ...)$
6.	l_2	$(\underbrace{\frac{1}{n^2}, \frac{1}{n^2}, ..., \frac{1}{n^2}}_{n}, n, 0, ...)$	$(0, 0, ..., 0, ...)$

Ex 5. Show that if a sequence $\{x_n\}_{n\in\mathbb{N}}$ in the space l_p, $p \in [1, \infty]$, is convergent to an element a then $\lim_{n\to\infty} x_n(k) = a(k)$ for all $k \in \mathbb{N}$.

Ex 6. Examine convergence of a sequence x_n in a space X.

N	X	x_n	N	X	x_n
1.	l_∞	$(\underbrace{tg(1+\frac{1}{n})^n, tg(1+\frac{1}{n})^n, ..., tg(1+\frac{1}{n})^n}_{n}, 0, 0, ...)$	6.	$l_{\frac{5}{2}}$	$((\frac{n+1}{n})^n, (\frac{n+2}{n})^n, ..., (\frac{n+(n-1)}{n})^n, 0, 0, ...)$
2.	l_3	$(\underbrace{\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, ..., \frac{1}{\sqrt{n}}}_{n}, 1, 0, ...)$	7.	c	$(\frac{1}{2}, \frac{4}{5}, ..., \frac{n^2}{n^2+1}, 0, 0, ...)$
3.	l_2	$(\underbrace{\sin\frac{1}{n}, \sin\frac{1}{n}, ..., \sin\frac{1}{n}}_{n^2}, 0, 0, ...)$	8.	l_1	$(\underbrace{\frac{\sin 3^n}{n^2}, \frac{\sin 3^n}{n^2}, ..., \frac{\sin 3^n}{n^2}}_{n}, 0, 0, ...)$
4.	c_0	$(tg(\frac{1}{n}), tg(\frac{1}{n^2}), ..., tg(\frac{1}{n^k}), tg(\frac{1}{n^{k+1}}), ...)$	9.	l_∞	$(1, \sqrt[2]{2}, \sqrt[3]{3}..., \sqrt[n]{n}, 0, 0, ...)$
5.	l_2	$(\underbrace{0, .., 0}_{n}, \underbrace{\frac{1}{\sqrt{n}}, ..., \frac{1}{\sqrt{n}}}_{n}, 0, 0, ...)$	10.	$l_{\sqrt{5}}$	$(\underbrace{\sin\frac{1}{\sqrt{n}}, \sin\frac{1}{\sqrt{n}}, ..., \sin\frac{1}{\sqrt{n}}}_{n}, 0, ...)$

Ex 7. Check whether a given subset M of l_p is open, closed or bounded.

N	p	M	N	p	M
1.	1	$\{x : \|x(k)\| \leq \frac{1}{k}\}$	3.	∞	$\{x : \exists_n \forall_{k>n} x(k) = 0\}$
2.	2	$\{x : x(k) > 0\}$	4.	1	$\{x : \sum_{k=1}^{\infty} \|x(k)\|^2 < 1\}$